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Dynamics in nonlocal linear models in the Friedmann–Robertson–Walker metric

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Abstract
A general class of cosmological models driven by a nonlocal scalar field inspired by string field theory is studied. Using the fact that the considering linear nonlocal model is equivalent to an infinite number of local models we have found an exact special solution of the nonlocal Friedmann equations. This solution describes a monotonically increasing universe with the phantom dark energy.

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1. Introduction

Recently, string theory and brane cosmology have been intensively discussed as promising candidates for the theoretical explanation of the obtained experimental data (see, for example, [1–6]).

The purpose of this paper is to present new results concerning studies of nonlocal linear models in the Friedmann–Robertson–Walker universe. These models are inspired by the string field theory (SFT) (for review of the SFT see [7]). A distinguished feature of nonlocal linear and nonlinear models [8–21] is the presence of infinite number of higher derivative terms (note also nonlocal models in the Minkowski spacetime [22–29]). For special values of the parameters these models describe linear approximations to the cubic bosonic or nonBPS fermionic SFT nonlocal tachyon models, p-adic string models or the models with the invariance of the action under the shift of the dilaton field to a constant. The nonBPS fermionic string field tachyon nonlocal model has been considered as a candidate for the dark energy [8].

Present cosmological observations [30] do not exclude an evolving dark energy (DE) state parameter w, whose current value can be less than -1, that means the violation of the null energy condition (NEC) (see [31, 32] for a review of the DE problems and [33] for a search for a super-acceleration phase of the universe).

Field theories, which violate the NEC [34, 35], are of interest not only for the construction of cosmological dark energy models with the state parameter $w < -1$, but also for the solution of the cosmological singularity problem. A possible way to avoid cosmological singularities consists of dealing with nonsingular bouncing cosmological solutions. In this scenario, the universe contracts before the bounce [2]. Such models have strong coupling and higher-order string corrections are inevitable. It is important to construct nonsingular bouncing cosmological solutions in order to make a concrete prediction of bouncing cosmology.

A simple possibility to violate the NEC is just to deal with a phantom field. In the present paper we consider nonlocal models which are linear and admit solutions, which are linear combinations of local fields. Some of these local fields are phantoms. Namely due to the presence of these ghost excitations such nonlocal models are of interest for cosmology.

At the same time there are well-known problems with instability of quantum models with phantoms, namely a loss of unitarity and so on. We believe that nonlocal SFT models in true vacua are stable with respect to quantum fluctuations. This question has to be considered in the full SFT framework and demands further investigations. We also believe that due to these string theory origins the corresponding nonlocal cosmological models, which are nonlinear in matter fields, have no problem with instability in the quantum case. In this paper we only consider the classical case and models, which are linear in a nonlocal scalar field.

In our previous paper [15] as well as in paper [16] nonlocal linear models have already been studied. In [15], the nonlocal linear model has been studied in the flat spacetime and we have proposed special deformations of the potential, which allow us to get the same scalar field solutions in flat and nonflat (the FRW metric) cases. As a result, we have obtained nonlinear models in the FRW metric. In [16] few exact solutions to the linear model in the FRW metric have been found. In this paper, we present a systematic method that permits us to transform the initial nonlocal system into an infinity set of local systems. The choice of a local system is equivalent to the choice of a special solution of the nonlocal system. This approach allows us to use the standard method of analysis of the differential equations and in particular to find exact solutions.

The paper is organized as follows. In section 2, we describe string-inspired models with quadratic nonlocal potentials. In section 3, we assume that the metric is given and consider the equation of motion as an equation for the nonlocal scalar field. We construct solutions, using eigenfunctions of the $\Box_g$-operator with eigenvalues, belonging to the set of roots of the characteristic equation. In section 4, we find values of the energy density and pressure for these solutions. In section 5, we consider the Friedmann–Robertson–Walker universe and find local models, which correspond to particular solutions of the initial nonlocal model. In the case of dilaton massless scalar field we construct the general solutions for the corresponding local model, which are the special exact solutions for the initial nonlocal model as well. We analyze cosmological properties of the obtained solutions.

2. Nonlocal linear models

In this paper we consider a model of gravity coupling with a nonlocal scalar field, which is induced by the string field theory

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_p^2}{2}R + \frac{M_s^4}{g_4}\left(\frac{1}{2}\phi F\left(-\Box_g/M_s^2\right)\phi - \Lambda'\right)\right), \qquad (1)$$

where $g_{\mu\nu}$ is the metric tensor (we always use the signature $(-,+,+,+)$), $\Box_g = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$, M_p is a mass Planck, M_s is a characteristic string scale related to the string tension α': $M_s = 1/\sqrt{\alpha'}$, ϕ is a dimensionless scalar field, g_4 is a dimensionless four-dimensional effective coupling constant related to the ten-dimensional string coupling constant g_0 and the compactification scale. $\Lambda = \frac{M_s^4}{g_4}\Lambda'$ is an effective four-dimensional cosmological constant.

The form of the function F is inspired by a nonlocal action appeared in the string field theory. We consider the case

$$F(z) = -\xi^2 z + 1 - c\,\mathrm{e}^{-2z}, \tag{2}$$

where ξ is a real parameter and c is a positive constant. Using dimensionless spacetime variables and a rescaling we can rewrite (1) for F given by (2) as follows:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{m_p^2}{2}R + \frac{\xi^2}{2}\phi\Box_g\phi + \frac{1}{2}(\phi^2 - c\,\Phi^2) - \Lambda'\right), \tag{3}$$

where

$$\Phi = \mathrm{e}^{\Box_g}\phi$$

and $m_p^2 = g_4 M_p^2 \big/ M_s^2$. Generally speaking the string scale does not coincide with the Planck mass. That gives a possibility to get a realistic value of Λ.

The form of the term $(\mathrm{e}^{\Box_g}\phi)^2$ is analogous to the form of the interaction term for the tachyon field in the SFT action. The case of the open cubic superstring field theory tachyon corresponds to $\xi^2 = -1\big/\left(4\ln\left(\frac{4}{3\sqrt{3}}\right)\right) \approx 0.9556$ and $c = 3$ (see [25–27]).

The equation of motion for the scalar field has the following form:

$$(\xi^2\Box_g + 1)\,\mathrm{e}^{-2\Box_g}\phi = c\,\phi. \tag{4}$$

The energy–momentum tensor

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} \tag{5}$$

has the following explicit form:

$$\begin{aligned} T_{\alpha\beta} = {} & -g_{\alpha\beta}\left(\frac{1}{2}\phi^2 - \frac{\xi^2}{2}\partial_\mu\phi\partial^\mu\phi - \frac{c}{2}(\mathrm{e}^{\Box_g}\phi)^2 - \Lambda'\right) - \xi^2\partial_\alpha\phi\partial_\beta\phi \\ & - g_{\alpha\beta}c\int_0^1 \mathrm{d}\rho\big[\big(\mathrm{e}^{(1+\rho)\Box_g}\phi\big)\big(\Box_g\,\mathrm{e}^{(1-\rho)\Box_g}\phi\big) + \big(\partial_\mu\,\mathrm{e}^{(1+\rho)\Box_g}\phi\big)\big(\partial^\mu\,\mathrm{e}^{(1-\rho)\Box_g}\phi\big)\big] \\ & + 2c\int_0^1 \mathrm{d}\rho\big(\partial_\alpha\,\mathrm{e}^{(1+\rho)\Box_g}\phi\big)\big(\partial_\beta\,\mathrm{e}^{(1-\rho)\Box_g}\phi\big). \end{aligned}$$

Note that the energy–momentum tensor $T_{\alpha\beta}$ includes the nonlocal terms, so the Einstein's equations are nonlocal ones.

3. Generalization of flat dynamics

3.1. Flat dynamics

In the flat case action (1) has the following form:

$$S_{\text{flat}} = \frac{1}{2}\int \mathrm{d}^4x\,\phi F(-\Box)\phi. \tag{6}$$

If the scalar field ϕ depends only on time, then the equation of motion (4) is reduced to the following linear equation:

$$F\left(\partial_0^2\right)\phi(t) = 0. \tag{7}$$

A plane wave $\phi = \mathrm{e}^{\alpha t}$ is a solution of (7) if α is a root of the characteristic equation

$$F(\alpha^2) = 0. \tag{8}$$

For the case of F given by (2) equation (7) has the following form:

$$-\xi^2\partial_0^2\phi + \phi - c\,\mathrm{e}^{-2\partial_0^2}\phi = 0. \tag{9}$$

This equation has been analyzed in detail in our paper [15]. Using the explicit form of the function $\phi(t)$ we have found the solutions of equations of motion and the corresponding values of the energy density and pressure. In this paper, we generalize these calculations for the nonflat case.

3.2. The equation of motion in an arbitrary metric

Let us consider equation (4). Really this equation is a consequence of the Einstein's equations, hence both the metric $g_{\mu\nu}$ and the scalar field ϕ are unknown. We assume that the metric $g_{\mu\nu}$ is given and consider equation (4) as an equation in ϕ.

In this paper we study solutions in the following form:

$$\phi = \sum_{n=1}^{N}\phi_n, \tag{10}$$

where N is a natural number, ϕ_n is a solution of the following equation:

$$\Box_g\phi_n = -\alpha_n^2\phi_n, \tag{11}$$

and α_n are solutions to the corresponding characteristic equation:

$$F\left(\alpha_n^2\right) \equiv -\xi^2\alpha_n^2 + 1 - c\,\mathrm{e}^{-2\alpha_n^2} = 0. \tag{12}$$

Without loss of generality we can assume that for any n and $k \neq n$ the conditions $\alpha_n^2 \neq \alpha_k^2$ are satisfied. Indeed, if the sum (10) includes two summands ϕ_{k_1} and ϕ_{k_2}, which correspond to one and the same α_k^2, then we can consider them as one summand $\phi_k \equiv \phi_{k_1} + \phi_{k_2}$, which corresponds to α_k^2.

We start with the construction of a solution to equation (4) in the case $N = 1$:

$$\Box_g\phi = -\alpha^2\phi, \tag{13}$$

where α is a root of (12). Note that this ansatz is widely used in studying nonlocal linear models [6, 13, 16–18, 36]. Equation (12) has the following solutions

$$\alpha_n = \pm\frac{1}{2\xi}\sqrt{4 + 2\xi^2 W_n\left(-\frac{2c\,\mathrm{e}^{-2/\xi^2}}{\xi^2}\right)}, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{14}$$

where W_n is the n–s branch of the Lambert function satisfying a relation $W(z)\,\mathrm{e}^{W(z)} = z$. The Lambert function is a multivalued function, so equation (12) has an infinite number of roots. Parameters ξ and c are real, therefore if α_n is a root of (12), then the adjoined number α_n^* is a root as well. Note that if α_n is a root of (12), then $-\alpha_n$ is a root too.

If $\alpha^2 = \alpha_0^2$ is a multiple root, then at this point $F\left(\alpha_0^2\right) = 0$ and $F'\left(\alpha_0^2\right) = 0$. These equations give that

$$\alpha_0^2 = \frac{1}{\xi^2} - \frac{1}{2}, \tag{15}$$

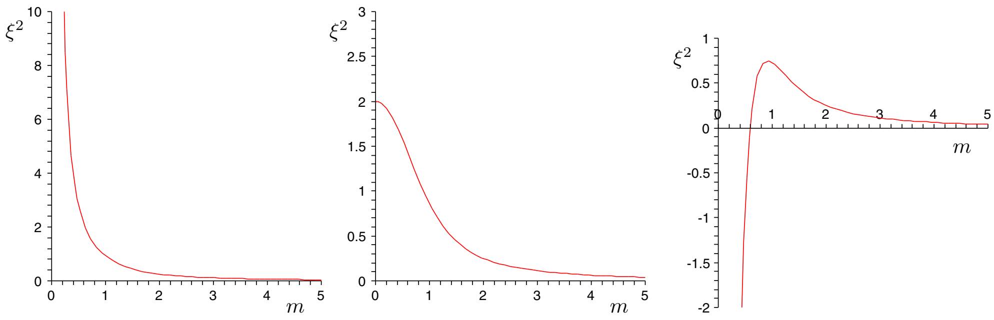

Figure 1. The dependence of the function $g(m^2, c)$, which is equal to ξ^2, on m at $c = 1/2$ (left), $c = 1$ (center) and $c = 2$ (right).

hence α_0^2 is a real number and all multiple roots of $F\left(\alpha_0^2\right) = 0$ are either real or pure imaginary. Double roots exist if and only if

$$c = \frac{\xi^2}{2}\,\mathrm{e}^{(2/\xi^2-1)}. \tag{16}$$

Note that the existence of double roots means that there exist solutions of equation (4), which do not satisfy equation (13), but satisfy the following equation:

$$\Box_g^2\phi = \alpha^4\phi. \tag{17}$$

In the flat case an example of such a solution is the function $\phi(t) = t\exp(\alpha t)$ (see [15]). All roots for any ξ and c are no more than double degenerated, because $F''\left(\alpha_0^2\right) \neq 0$. In this paper, we consider such values of ξ and c that equality (16) is not satisfied and all roots are simple ones. Under this assumption we can consider the set of the solutions (10) as a quite general solution.

3.3. Real roots of the characteristic equation

For some values of the parameters ξ and c equation (12) has real roots. To mark out real values of α we will denote real α as m: $m = \alpha$.

To determine values of the parameters at which equation (12) has real roots we rewrite this equation in the following form:

$$\xi^2 = g(m^2, c), \qquad \text{where} \quad g(m^2, c) = \frac{\mathrm{e}^{2m^2} - c}{m^2\,\mathrm{e}^{2m^2}}. \tag{18}$$

The dependence of $g(m, c)$ on m for different c is presented in figure 1. This function has a maximum at $m_{\max}^2$

$$m_{\max}^2 = -\frac{1}{2} - \frac{1}{2}W_{-1}\left(-\frac{\mathrm{e}^{-1}}{c}\right), \tag{19}$$

provided c is such that $W_{-1}\left(-\frac{\mathrm{e}^{-1}}{c}\right) < -1$, in other words $0 < c < 1$.

There are three different cases (see figure 1):

- If $c < 1$, then equation (12) has two simple real roots: $m = \pm m_1$ for any values ξ.
- If $c = 1$, then equation (12) has a zero root. Nonzero real roots exist if and only if $\xi^2 < 2$.
- If $c > 1$, then equation (12) has

– no real roots for $\xi^2 > \xi^2_{\max}$, where

$$\xi^2_{\max} = \frac{1 - c\,\mathrm{e}^{-2m^2_{\max}}}{m^2_{\max}} = -\frac{2}{W_{-1}(-\mathrm{e}^{-1}/c)}; \tag{20}$$

– two real double roots $m = \pm m_{\max}$ for $\xi^2 = \xi^2_{\max}$;
– four real simple roots for $\xi^2 < \xi^2_{\max}$. In this case we have the following restriction on real roots: $m^2 > \frac{1}{2}\ln c$.

Note that the values of roots do not depend on $H(t)$ and, therefore, coincide with roots in the flat case, which have been found in [15].

4. Energy density and pressure

4.1. General formula

Let us calculate the energy density and the pressure for the solution (10). Up to this moment we do not put any restrictions on the metric tensor $g_{\mu\nu}$, now we start to consider the case of the spatially flat Friedmann–Robertson–Walker universe:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right) \tag{21}$$

and spatially homogeneous solutions $\phi(t)$. In this case,

$$T_{\alpha\beta} = g_{\alpha\beta}\,\mathrm{diag}\{\mathcal{E}, -\mathcal{P}, -\mathcal{P}, -\mathcal{P}\}, \tag{22}$$

where the energy density $\mathcal{E}$ and pressure $\mathcal{P}$ are as follows:

$$\mathcal{E} = \mathcal{E}_k + \mathcal{E}_p + \mathcal{E}_{nl2} + \mathcal{E}_{nl1} + \Lambda', \qquad \mathcal{P} = \mathcal{E}_k - \mathcal{E}_p + \mathcal{E}_{nl2} - \mathcal{E}_{nl1} - \Lambda'. \tag{23}$$

Nonlocal term $\mathcal{E}_{nl1}$ plays the role of an extra potential term and $\mathcal{E}_{nl12}$ plays the role of an extra kinetic term. The explicit form of the terms on the rhs of (23) is [24, 29] as follows:

$$\begin{aligned}
\mathcal{E}_k &= \frac{\xi^2}{2}(\partial_0\phi)^2,\\
\mathcal{E}_p &= -\frac{1}{2}(\phi^2 - c(\mathrm{e}^{\mathcal{D}}\phi)^2),\\
\mathcal{E}_{nl1} &= c\int_0^1 (\mathrm{e}^{(1+\rho)\mathcal{D}}\phi)(-\mathcal{D}\,\mathrm{e}^{(1-\rho)\mathcal{D}}\phi)\,\mathrm{d}\rho,\\
\mathcal{E}_{nl2} &= -c\int_0^1 (\partial\,\mathrm{e}^{(1+\rho)\mathcal{D}}\phi)(\partial\,\mathrm{e}^{(1-\rho)\mathcal{D}}\phi)\,\mathrm{d}\rho,
\end{aligned} \tag{24}$$

where

$$\mathcal{D} \equiv -\partial_0^2 - 3H(t)\partial_0, \qquad H = \frac{\partial_0 a}{a}. \tag{25}$$

For $N = 1$ we obtain

$$\mathcal{E} \equiv E(\phi_1) + \Lambda' = \frac{\eta_{\alpha_1}}{2}\left((\partial_0\phi_1)^2 - \alpha_1^2\phi_1^2\right) + \Lambda', \tag{26}$$

$$\mathcal{P} \equiv P(\phi_1) - \Lambda' = \frac{\eta_{\alpha_1}}{2}\left((\partial_0\phi_1)^2 + \alpha_1^2\phi_1^2\right) - \Lambda', \tag{27}$$

where for arbitrary α

$$\eta_\alpha \equiv \xi^2 + 2\xi^2\alpha^2 - 2. \tag{28}$$

Note that considering the flat spacetime [15], we have introduced the parameter $p_\alpha \equiv \alpha^2\eta_\alpha$. The use of parameter η_α instead of p_α is more convenient, because we do not need to consider the case $\alpha = 0$ separately.

Hereafter, we denote the energy density and pressure of the function $\phi(t)$ as the functionals $E(\phi)$ and $P(\phi)$, respectively.

For the solution $\phi(t) = \phi_1(t) + \phi_2(t)$ it is convenient to write the energy density in the following form:

$$\mathcal{E} = E(\phi_1 + \phi_2) + \Lambda' = E(\phi_1) + E(\phi_2) + E_{\text{cross}}(\phi_1, \phi_2) + \Lambda',$$

where the functional $E_{\text{cross}}(\phi_1, \phi_2)$ is defined as follows:

$$\begin{aligned}
&E_{\text{cross}}(\phi_1, \phi_2) = E_{k_{\text{cr}}} + E_{nl2_{\text{cr}}} + E_{p_{\text{cr}}} + E_{nl1_{\text{cr}}},\\
&E_{k_{\text{cr}}} \equiv \xi^2 \partial_0\phi_1 \partial_0\phi_2, \qquad E_{p_{\text{cr}}} \equiv -\phi_1\phi_2 + c\,\mathrm{e}^{-\alpha_1^2-\alpha_2^2}\phi_1\phi_2,\\
&E_{nl1_{\text{cr}}} \equiv -c\int_0^1 \left[\left(\mathrm{e}^{(1+\rho)\mathcal{D}}\phi_1\right)\mathcal{D}\left(\mathrm{e}^{(1-\rho)\mathcal{D}}\phi_2\right) + \left(\mathrm{e}^{(1+\rho)\mathcal{D}}\phi_2\right)\mathcal{D}\left(\mathrm{e}^{(1-\rho)\mathcal{D}}\phi_1\right)\right] \mathrm{d}\rho,\\
&E_{nl2_{\text{cr}}} \equiv -c\int_0^1 \left[\partial_0\left(\mathrm{e}^{(1+\rho)\mathcal{D}}\phi_1\right)\partial_0\left(\mathrm{e}^{(1-\rho)\mathcal{D}}\phi_2\right) + \partial_0\left(\mathrm{e}^{(1+\rho)\mathcal{D}}\phi_2\right)\partial_0\left(\mathrm{e}^{(1-\rho)\mathcal{D}}\phi_1\right)\right] \mathrm{d}\rho.
\end{aligned} \tag{29}$$

Using (12), we calculate $E_{nl2_{\text{cr}}}$:

$$E_{nl2_{\text{cr}}} = -\frac{c\left(\mathrm{e}^{-2\alpha_1^2} - \mathrm{e}^{-2\alpha_2^2}\right)}{\alpha_2^2 - \alpha_1^2}\partial_0\phi_1\partial_0\phi_2 = -\xi^2\partial_0\phi_1\partial_0\phi_2. \tag{30}$$

So,

$$E_{nl2_{\text{cr}}} + E_{k_{\text{cr}}} = 0. \tag{31}$$

The straightforward calculation also gives that

$$E_{nl1_{\text{cr}}} = -c\,\mathrm{e}^{-\alpha_1^2-\alpha_2^2}\phi_1\phi_2 + \frac{c\left(\alpha_2^2\,\mathrm{e}^{-2\alpha_1^2} - \alpha_1^2\,\mathrm{e}^{-2\alpha_2^2}\right)}{\alpha_2^2 - \alpha_1^2}\phi_1\phi_2 = -E_{p_{\text{cr}}}. \tag{32}$$

Therefore, we obtain that

$$E_{\text{cross}}(\phi_1, \phi_2) = 0 \qquad \text{and} \qquad P_{\text{cross}}(\phi_1, \phi_2) = 0, \tag{33}$$

where

$$P_{\text{cross}}(\phi_1, \phi_2) \equiv E_{k_{\text{cr}}} + E_{nl2_{\text{cr}}} - E_{p_{\text{cr}}} - E_{nl1_{\text{cr}}}. \tag{34}$$

So,

$$E(\phi_1 + \phi_2) = E(\phi_1) + E(\phi_2), \tag{35}$$

$$P(\phi_1 + \phi_2) = P(\phi_1) + P(\phi_2). \tag{36}$$

Finally, for the case of N summands we obtain (compare with [14, 15, 17])

$$\mathcal{E} = E\left(\sum_{n=1}^{N}\phi_n\right) + \Lambda' = \sum_{n=1}^{N} E(\phi_n) + \Lambda', \tag{37}$$

$$\mathcal{P} = P\left(\sum_{n=1}^{N}\phi_n\right) - \Lambda' = \sum_{n=1}^{N} P(\phi_n) - \Lambda'. \tag{38}$$

From formulae (37) and (38) we see that the energy density and the pressure are sums of 'individual' energy densities and pressures, respectively, and have no crossing term.

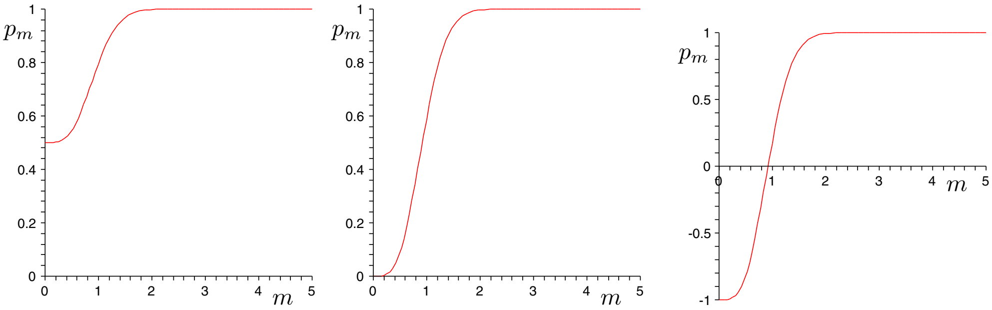

Figure 2. The dependence of p_m on m at $c = 1/2$ (right), $c = 1$ (center) and $c = 2$ (left).

In the case of an arbitrary metric $g_{\alpha\beta}$ and a scalar field $\phi_n(t, x_1, x_2, x_3)$, which satisfies equation (11), we obtain that

$$\begin{aligned}
T_{\alpha\beta}(\phi_n) &= -g_{\alpha\beta}\left(\frac{1}{2}\phi_n^2 - \frac{\xi^2}{2}\partial_\mu\phi_n\partial^\mu\phi_n - \frac{c}{2}\left(\mathrm{e}^{\Box_g}\phi_n\right)^2\right) - \xi^2\partial_\alpha\phi_n\partial_\beta\phi_n \\
&\quad - c\, g_{\alpha\beta}\int_0^1 \mathrm{d}\rho\left[\left(\mathrm{e}^{(1+\rho)\Box_g}\phi_n\right)\left(\Box_g\,\mathrm{e}^{(1-\rho)\Box_g}\phi_n\right) + \left(\partial_\mu\,\mathrm{e}^{(1+\rho)\Box_g}\phi_n\right)\left(\partial^\mu\,\mathrm{e}^{(1-\rho)\Box_g}\phi_n\right)\right] \\
&\quad + 2c\int_0^1 \mathrm{d}\rho\left(\partial_\alpha\,\mathrm{e}^{(1+\rho)\Box_g}\phi_n\right)\left(\partial_\beta\,\mathrm{e}^{(1-\rho)\Box_g}\phi_n\right) \\
&= g_{\alpha\beta}\left(\frac{\eta_{\alpha_n}}{2}\partial_\mu\phi_n\partial^\mu\phi_n - \frac{\eta_{\alpha_n}\alpha_n^2}{2}\phi_n^2\right) - \eta_{\alpha_n}\partial_\alpha\phi_n\partial_\beta\phi_n.
\end{aligned}$$

The energy–momentum tensor, which corresponds to the function (10), is as follows:

$$T_{\alpha\beta} = T_{\alpha\beta}\left(\sum_{n=1}^{N}\phi_n\right) + g_{\alpha\beta}\Lambda' = \sum_{n=1}^{N}T_{\alpha\beta}(\phi_n) + g_{\alpha\beta}\Lambda'. \tag{39}$$

4.2. *Energy density and pressure for real α*

As we have seen in section 3 for some values of parameters ξ and c equation (12) has real roots. We denote as η_m the value of η_α for real $\alpha = m$:

$$\eta_m = \xi^2(1 + 2m^2) - 2 = \frac{\mathrm{e}^{2m^2} - c}{m^2\,\mathrm{e}^{2m^2}}(1 + 2m^2) - 2. \tag{40}$$

If and only if $c > 1$, then there exists the interval $0 < m^2 < m_{\max}^2$, on which $\eta_m < 0$. Some part of this interval is not physical, because $g(m^2, c) < 0$ on this part. The straightforward calculations (compare with [15]) show that at the point

$$m_{\max}^2 = -\frac{1}{2} - \frac{1}{2}W_{-1}\left(-\frac{\mathrm{e}^{-1}}{c}\right) \tag{41}$$

we obtain $\eta_m(m_{\max}) = 0$. So, for $c > 1$ and $\xi^2 < \xi_{\max}^2$ we have two positive roots of (12): m_1 and $m_2 > m_1$, with $\eta_{m_1} < 0$ and $\eta_{m_2} > 0$. In the following section we use this fact to construct a quintom local model with one tachyon real scalar field, which corresponds to η_{m_2}, and one phantom real scalar field, which corresponds to η_{m_1}. For different values of c the function $p_m \equiv m^2\eta_m$ is presented in figure 2.

5. Construction of solutions in the Friedmann–Robertson–Walker metric

5.1. Equations of motion and Friedmann equations

In the spatially flat Friedmann–Robertson–Walker universe we get the following equation of motion for the space homogeneous scalar field ϕ:

$$(\xi^2\mathcal{D}+1)\,\mathrm{e}^{-2\mathcal{D}}\phi = c\,\phi. \tag{42}$$

The Friedmann equations have the following form:

$$\begin{cases} 3H^2 = \dfrac{1}{m_p^2}\mathcal{E}, \\ \dot{H} = -\dfrac{1}{2m_p^2}(\mathcal{E}+\mathcal{P}), \end{cases} \tag{43}$$

where a dot denotes the time derivative ($\dot{H} \equiv \partial_0 H$).

The second equation of system (43) is the nonlinear integral equation in $H(t)$:

$$\dot{H} = -\frac{1}{m_p^2}\left(\frac{\xi^2}{2}(\partial_0\phi)^2 - c\int_0^1 (\partial_0\,\mathrm{e}^{(1+\rho)\mathcal{D}}\phi)(\partial_0\,\mathrm{e}^{(1-\rho)\mathcal{D}}\phi)\,\mathrm{d}\rho\right). \tag{44}$$

Let us make an assumption that $\phi(t)$ and $H(t)$ satisfy the following equation

$$\mathcal{D}\phi = -\alpha^2\phi, \tag{45}$$

where α is a root of equation (12).

In this case equation (42) is solved. Using formulae (26) and (27), we rewrite system (43) in the following form:

$$\begin{cases} 3H^2 = \dfrac{\eta_\alpha}{2m_p^2}(\dot{\phi}^2 - \alpha^2\phi^2 + \Lambda'), \\ \dot{H} = -\dfrac{\eta_\alpha}{2m_p^2}\dot{\phi}^2. \end{cases} \tag{46}$$

It is easy to check that (45) is a consequence of system (46). Instead of (46) we can consider the following third-order system:

$$\begin{cases} \ddot{\phi} + 3H\dot{\phi} = \alpha^2\phi, \\ \dot{H} = -\dfrac{\eta_\alpha}{2m_p^2}\dot{\phi}^2. \end{cases} \tag{47}$$

This system has the following integral of motion:

$$I_1 = 3H^2 - \frac{\eta_\alpha}{2m_p^2}(\dot{\phi}^2 - \alpha^2\phi^2) = \frac{\eta_\alpha}{2m_p^2}\Lambda', \tag{48}$$

therefore, choosing the initial data for (47) one fixes the value of Λ'.

So, our assumption allows us to transform a system with a nonlocal scalar field into a system with a local one. In the same way, we obtain systems with two or more local fields. Let

$$\phi(t) = \sum_{n=1}^{N}\phi_n(t), \tag{49}$$

where all $\phi_n(t)$ are solutions of (45) with the same function $H(t)$ and different values of α: $\alpha = \alpha_n$. If all α_n $(n = 1 \ldots N)$ are different roots of (12), then system (43) transforms into the following system with N scalar fields:

$$\begin{cases} 3H^2 = \dfrac{1}{2m_p^2}\left(\displaystyle\sum_{n=1}^{N} \eta_{\alpha_n}\left(\dot\phi_n^2 - \alpha_n^2\phi_n^2\right) + \Lambda'\right), \\ \dot H = -\dfrac{1}{2m_p^2}\left(\displaystyle\sum_{n=1}^{N} \eta_{\alpha_n}\dot\phi_n^2\right). \end{cases} \tag{50}$$

In the case of two real roots $\alpha_1 > 0$ and $\alpha_2 > \alpha_1$:

$$\begin{cases} 3H^2 = \dfrac{1}{2m_p^2}\left(\eta_{\alpha_1}\left(\dot\phi_1^2 - \alpha_1^2\phi_1^2\right) + \eta_{\alpha_2}\left(\dot\phi_2^2 - \alpha_2^2\phi_2^2\right) + \Lambda'\right), \\ \dot H = -\dfrac{1}{2m_p^2}\left(\eta_{\alpha_1}\dot\phi_1^2 + \eta_{\alpha_2}\dot\phi_2^2\right), \end{cases} \tag{51}$$

we have obtained that $\eta_{\alpha_1} < 0$ and $\eta_{\alpha_2} > 0$. Therefore the corresponding two-field model is a quintom one, in other words, includes one phantom scalar field ($\eta_{\alpha_1} < 0$) and one scalar field with the canonical kinetic term ($\eta_{\alpha_2} > 0$) and with the tachyon mass term $\left(\alpha_2^2\eta_{\alpha_2} > 0\right)$. The SFT-inspired nonlinear local quintom models and their exact solutions have been studied, for example, in [37, 38]. To obtain exact solutions with physically important properties usually one should add some additional terms in the potential, which tend to zero in the limit of the flat spacetime [15, 37–39]. It is interesting that system (46) allows us to find a physically important exact solution without adding any term in the potential.

5.2. *Exact solution in the case $N = 1$*

Let us consider system (46) with real α. Two exact nontrivial real solutions of this system have been presented in [16]. In our notations these solutions are the following:

- At $\alpha \neq 0$ and $\eta_\alpha < 0$

$$\phi(t) = A(t - t_0), \qquad \Lambda' = -A^2, \qquad H(t) = \frac{\alpha^2}{3}(t - t_0), \tag{52}$$

  where

$$A = \pm\sqrt{-\frac{2m_p^2\alpha^2}{3\eta_\alpha}}, \tag{53}$$

  t_0 is an arbitrary constant.
- At $\alpha = 0$, $\Lambda' = 0$ and $\eta_\alpha = \xi^2 - 2 > 0$

$$\phi(t) = \pm\sqrt{\frac{2m_p^2}{3\eta_\alpha}}\ln(t - t_0) + C_1, \qquad H(t) = \frac{1}{3(t - t_0)}, \tag{54}$$

  where t_0 and C_1 are arbitrary constants. Note that the root $\alpha = 0$ exists if and only if $c = 1$.

In this paper we present a new solution, which looks more realistic for the SFT-inspired cosmological model. At the present time, one of the possible scenarios of the universe evolution considers the universe to be a D3-brane (three spatial and one time variable) embedded in higher-dimensional spacetime. This D-brane is unstable and does evolve to the stable state. This process is described by the dynamics of the open string, whose ends are attached to the

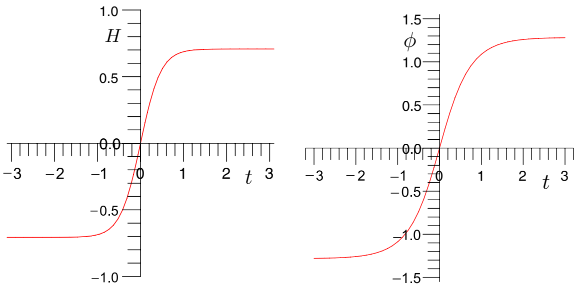

Figure 3. The functions $H_1(t)$ (right) and $\phi_1(t)$ (left) at $\Lambda' = 3, m_p^2 = 1, \xi^2 = 1, t_0 = 0$ and $C_2 = 0$.

brane (see reviews [7] and references therein). A phantom scalar field is an open string theory tachyon. According to the Sen's conjecture [28], this tachyon describes brane decay, at which a slow transition in a stable vacuum takes place. This vacuum is characterized by the absence of open string states, i.e. corresponds to states of the closed string. This picture allows us to specify the asymptotic conditions for the scalar field. We assume that the phantom field $\phi(t)$ smoothly rolls from the unstable perturbative vacuum ($\phi = 0$) to a nonperturbative one, for example $\phi = A_0$, where A_0 is a nonzero constant and stops there. It is easy to see that exact solutions presented in [16] do not satisfy these conditions.

At $c = 1$ our model (3) is a nonlocal model for the dilaton coupling to the gravitation field. Its distinguished feature is the invariance under the shift of the dilaton field to a constant. In this case one of the solutions of equation (12) is $\alpha = 0$. Summing the first and second equations of (46), we obtain

$$\dot{H} = \frac{\Lambda'}{2m_p^2} - 3H^2. \tag{55}$$

If $\Lambda' > 0$, then we obtain a real solution:

$$H_1(t) = \sqrt{\frac{\Lambda'}{6m_p^2}} \tanh\left(\sqrt{\frac{3\Lambda'}{2m_p^2}}(t - t_0)\right), \tag{56}$$

where t_0 is an arbitrary real constant.

It is easy to see that $\dot{H}_1(t) > 0$ for any t, hence from the second equation of (46) we obtain that $\phi(t)$ can be a real scalar field only if it is a phantom one ($\eta_\alpha < 0$, that is equivalent to $\xi^2 < 2$). The explicit form of $\phi(t)$ is as follows:

$$\phi_1(t) = \pm\sqrt{\frac{2m_p^2}{3(2-\xi^2)}} \arctan\left(\sinh\left(\sqrt{\frac{3\Lambda'}{2m_p^2}}(t - t_0)\right)\right) + C_2, \tag{57}$$

where C_2 is an arbitrary constant. Functions $H_1(t)$ and $\phi_1(t)$ are presented in figure 3.

The Hubble parameter $H_1(t)$ is a monotonically increasing function, so using

$$w = -1 - \frac{2}{3}\frac{\dot{H}_1}{H_1^2}, \tag{58}$$

we obtain $w < -1$. So, solution (56) corresponds to phantom dark energy. Note that we have found two-parameter set of exact solutions at any $\Lambda' > 0$. In other words, at any $\Lambda' > 0$ we have found the general solution of (46), which corresponds to $\alpha = 0$. At $\Lambda' = 0$ the solution (56) transforms to a constant. In the case $\Lambda' = 0$ the general solution has been found in [16].

In the case $\Lambda' < 0$ we obtain the following general solution:

$$H_2(t) = -\sqrt{\frac{-\Lambda'}{6m_p^2}}\tan\left(\sqrt{-\frac{3\Lambda'}{2m_p^2}}(t-t_0)\right), \tag{59}$$

$$\phi_2(t) = \pm\sqrt{\frac{8(\xi^2-2)}{3m_p^2}}\operatorname{arctanh}\left(\frac{\cos\left(\sqrt{\frac{-3\Lambda'}{2m_p^2}}(t-t_0)\right)-1}{\sin\left(\sqrt{\frac{-3\Lambda'}{2m_p^2}}(t-t_0)\right)}\right)+C_2. \tag{60}$$

This solution is real at $\xi^2 > 2$. It is interesting that the type of solutions essentially depends on the sign of Λ'. The solution with the SFT-inspired boundary conditions corresponds to $\Lambda' > 0$.

6. Conclusions

We have studied the SFT-inspired linear nonlocal model. This model has an infinite number of higher derivative terms and are characterized by two positive parameters: ξ^2 and c. For particular cases of the parameters ξ^2 and c the corresponding actions describe linear approximations to either the bosonic or nonBPS fermionic cubic SFT as well as to the nonpolynomial SFT.

Roots of the characteristic equation do not depend on the form of the metric and this property allows us to study the properties of the energy density and pressure. We have found that in an arbitrary metric the energy–momentum tensor for an arbitrary N-mode solution is a sum of the energy–momentum tensors for the corresponding one-mode solutions. In the Friedmann–Robertson–Walker spatially flat metric the pressure for a one-mode solution corresponding to a real root can be positive or negative, depending on parameters of our nonlocal model. Namely, for $c \leqslant 1$ the one-mode pressure is positive and for $c > 1$ it could be negative or positive.

The investigation performed in this paper shows that the general field equations in linear nonlocal models admit an equivalent description in terms of local theory and as a consequence we have representations (37) and (38) for the energy and pressure. This calculation also supports the use of the Ostrogradski representation for our system in the case of arbitrary metric.

To distinguish our previous paper [15] we do not use any approximation scheme and do not add any terms in the potential. We have shown that our linear model with one nonlocal scalar field generates an infinite number of local models. These models can be studied numerically and we plan to present this analysis in future papers. Some of these models have been solved explicitly and, hence, special exact solutions for the nonlocal model in the Friedmann–Robertson–Walker metric have been obtained. In particular we have constructed an exact kink-like solution, which corresponds to monotonically increasing universe with phantom dark energy. Note that the obtained behavior of the Hubble parameter is close to the behavior of the Hubble parameter in the nonlinear nonlocal model [8], which has recently been obtained numerically [19].

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